A proof of uniqueness of the Gurariĭ space *

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Abstract

We present a short and elementary proof of isometric uniqueness of the Gurariĭ space.

1 Introduction

A *Gurariĭ space*, constructed by Gurariĭ [3] in 1965, is a separable Banach space $\mathbb{G}$ satisfying the following condition: given finite dimensional Banach spaces $X \subseteq Y$, given $\varepsilon > 0$, and given an isometric linear embedding $f \colon X \to \mathbb{G}$ there exists an injective linear operator $g \colon Y \to \mathbb{G}$ extending f and satisfying $\|g\| \cdot \|g^{-1}\| < 1 + \varepsilon$. It is not hard to prove straight from this definition that such a space is unique up to isomorphism of norm arbitrarily close to one. The question whether the Gurariĭ space is unique up to isometry remained open for some time. It was answered affirmatively by Lusky [6] in 1976 using deep techniques developed by Lazar and Lindenstrauss [5]. Subsequently, another proof of uniqueness was given by Henson using model theoretic methods of continuous logic. (This proof remains unpublished.) The natural question whether there is an elementary proof of uniqueness occurred to several mathematicians. This question was made current by recent increased interest in universal homogeneous structures and their automorphism groups; see, for example, [4] and [7]. The aim of this note is to provide just such a simple and elementary proof of isometric uniqueness of the Gurariĭ space. This proof is given in Section 2. In Section 3, we give an elementary proof showing isometric universality of the Gurariĭ space among separable Banach spaces. Our argument uses only the basic Gurariĭ property.

In order to state the theorem precisely, we introduce some notions. Let X, Y be Banach spaces, $\varepsilon > 0$. A linear operator $f \colon X \to Y$ is an *ε-isometry* if for $x \in X$ with $\|x\| = 1$

$$(1 + \varepsilon)^{-1} < \|f(x)\|\| < 1 + \varepsilon.$$

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We use strict inequalities for the sake of convenience. In particular, in the case of finite dimensional spaces, every ε-isometry is an ε'-isometry for some $0 < \varepsilon' < \varepsilon$. Note that the inverse of a bijective ε-isometry is again an ε-isometry. By an *isometry* we mean a linear operator $f\colon X \to Y$ that is an ε-isometry for every $\varepsilon > 0$, that is, $\|f(x)\| = \|x\|$ holds for every $x \in X$. (A word of caution about our terminology may be in place: in the literature, such functions are often called *isometric embeddings*, with the word "isometry" reserved for a *bijective* isometric embedding.)

We will give a proof of the following theorem.

Theorem 1.1. *Let E, F be separable Gurariĭ spaces, $\varepsilon > 0$. Assume $X \subseteq E$ is a finite dimensional space and $f\colon X \to F$ is an ε-isometry. Then there exists a bijective isometry $h\colon E \to F$ such that $\|h \restriction X - f\| < \varepsilon$.*

By taking X to be the trivial space, we obtain the following corollary.

Corollary 1.2 (Lusky [6])**.** *The Gurariĭ space is unique up to a bijective isometry.*

2 Proof of uniqueness of the Gurariĭ space

Lemma 2.1. *Let X, Y be finite dimensional Banach spaces and let $f\colon X \to Y$ be an ε-isometry, for some $\varepsilon > 0$. There exist a finite dimensional Banach space Z and isometries $i\colon X \to Z$ and $j\colon Y \to Z$ such that $\|j \circ f - i\| \le \varepsilon$.*

Proof. By $\|\cdot\|_X$, $\|\cdot\|_Y$ we denote the norms of X and Y, respectively.

First assume that f is onto, that is, $f[X] = Y$. Consider the space $X \oplus Y$ and the canonical embeddings $i\colon X \to X \oplus Y$ and $j\colon Y \to X \oplus Y$. We aim to define a suitable norm $\|\cdot\|'$ on $X \oplus Y$. With $x^* \in S_X^*$ associate the functional $\overline{x}^* = x^* f^{-1}$ on Y. Define

$$\varphi_X(x,y) = \sup_{x^* \in S_X^*} \left| x^*(x) + \frac{1}{\|\overline{x}^*\|_Y^*} \overline{x}^*(y) \right|.$$

It is clear that φ_X is a seminorm on $X \oplus Y$. Observe that $\varphi_X(x,0) = \|x\|_X$ and $\varphi_X(0,y) \le \|y\|_Y$. Interchanging the roles of X and Y and of $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ define in an analogous way φ_Y so that it is a seminorm on $X \oplus Y$ such that $\varphi_Y(x,0) \le \|x\|_X$ and $\varphi_Y(0,y) = \|y\|_Y$. Finally, define

$$\|(x,y)\|' = \max\Big\{\varphi_X(x,y), \varphi_Y(x,y), \varepsilon_1\|x\|_X, \varepsilon_1\|y\|_Y\Big\},$$

where $\varepsilon_1 = \frac{\varepsilon}{1+\varepsilon}$. Now $\|\cdot\|'$ is a norm on $X \oplus Y$ and, since $\varepsilon_1 \le 1$, we have that $\|(x,0)\|' = \|x\|_X$ and $\|(0,y)\|' = \|y\|_Y$. Hence, i and j are isometries.

We check that $\|jf(x)-i(x)\|' \le \varepsilon$ for $x \in S_X$. Set $u = jf(x)-i(x) = (-x, f(x))$. From the inequalities $(1+\varepsilon)^{-1} < \|x^* f^{-1}\|_Y^* < 1+\varepsilon$ for $x^* \in S_X^*$, we obtain

$$\begin{aligned}\varphi_X(u) &= \sup_{x^*\in S_X^*}\left|x^*(-x) + \frac{1}{\|\overline{x}^*\|_Y^*}\overline{x}^*(f(x))\right| = \sup_{x^*\in S_X^*}\left(\left|\frac{1}{\|x^* f^{-1}\|_Y^*} - 1\right| \cdot |x^*(x)|\right)\\ &\le \sup_{x^*\in S_X^*}\left|\frac{1}{\|x^* f^{-1}\|_Y^*} - 1\right| \le \varepsilon.\end{aligned}$$

Similarly, from $(1+\varepsilon)^{-1} < \|y^* f\|_X^* < 1+\varepsilon$ for $y^* \in S_Y^*$, we get

$$\begin{aligned}\varphi_Y(u) &= \sup_{y^*\in S_Y^*}\left|y^*(f(x)) + \frac{1}{\|\overline{y}^*\|_X^*}\overline{y}^*(-x)\right| = \sup_{y^*\in S_Y^*}\left(\left|1 - \frac{1}{\|y^* f\|_X^*}\right| \cdot |y^*(f(x))|\right)\\ &\le \sup_{y^*\in S_Y^*}\left(\left|1 - \frac{1}{\|y^* f\|_X^*}\right| \cdot \|y^* f\|_X^*\right) = \sup_{y^*\in S_Y^*} |\|y^* f\|_X^* - 1| \le \varepsilon.\end{aligned}$$

Finally, since

$$\varepsilon_1\|-x\|_X \le \varepsilon \ \text{ and } \ \varepsilon_1\|f(x)\|_Y \le \frac{\varepsilon}{1+\varepsilon}(1+\varepsilon) = \varepsilon,$$

we conclude that $\|u\|' \le \varepsilon$, as required.

Now we consider the general case when f is not necessarily onto. The conclusion above gives a norm $\|\cdot\|'$ on $X \oplus f[X]$. Take $(X \oplus f[X]) \oplus Y$ regarded as the ℓ_1 sum of the Banach spaces $(X \oplus f[X], \|\cdot\|')$ and $(Y, \|\cdot\|_Y)$. Pass to the quotient Banach space

$$Z = ((X \oplus f[X]) \oplus Y)/\{(0, f(v), -f(v)) \colon v \in X\}$$

with the quotient norm and with the canonical embeddings of X and Y. This space is as required. Note that Z is canonically isometric to $X \oplus Y$ equipped with the norm

$$\|(x,y)\| = \inf_{v\in X}\Big(\|(x, f(v))\|' + \|y - f(v)\|_Y\Big). \qquad \square$$

Lemma 2.2. *Let E be a Gurariĭ space and let $f\colon X \to Y$ be an ε-isometry, where X is a finite dimensional subspace of E and $\varepsilon > 0$. Then for every $\delta > 0$ there exists a δ-isometry $g\colon Y \to E$ such that $\|g \circ f - \mathrm{id}_X\| < \varepsilon$.*

Proof. Choose $0 < \varepsilon' < \varepsilon$ so that f is an ε'-isometry. Choose $0 < \delta' < \delta$ such that $(1+\delta')\varepsilon' < \varepsilon$. By Lemma 2.1, there exist a finite dimensional space Z and isometries $i\colon X \to Z$ and $j\colon Y \to Z$ satisfying $\|j \circ f - i\| \le \varepsilon'$. Since E is Gurariĭ, there exists a δ'-isometry $h\colon Z \to E$ such that $hj(x) = x$ for $x \in X$. Let $g = h \circ j$. Clearly, g is a δ-isometry. Finally, given $x \in S_X$, we have

$$\|gf(x) - x\| = \|hjf(x) - hi(x)\| < (1+\delta')\|jf(x) - i(x)\| \le (1+\delta')\varepsilon' < \varepsilon,$$

as required. $\square$

Proof of Theorem 1.1. Fix a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers. The precise conditions on $\{\varepsilon_n\}_{n\in\mathbb{N}}$ will be specified later. We define inductively sequences of linear operators $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ and finite dimensional subspaces $\{X_n\}_{n\in\mathbb{N}}$, $\{Y_n\}_{n\in\mathbb{N}}$ of E and F, respectively, so that the following conditions are satisfied:

(0) $X_0 = X$, $Y_0 = f[X]$, and $f_0 = f$;

(1) $f_n \colon X_n \to Y_n$ is an ε_n-isometry;

(2) $g_n \colon Y_n \to X_{n+1}$ is an ε_{n+1}-isometry;

(3) $\|g_n f_n(x) - x\| \leq \varepsilon_n \|x\|$ for $x \in X_n$;

(4) $\|f_{n+1} g_n(y) - y\| \leq \varepsilon_{n+1}\|y\|$ for $y \in Y_n$;

(5) $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$, $\bigcup_n X_n$ and $\bigcup_n Y_n$ are dense in E and F, respectively.

Condition (0) tells us how to start the inductive construction. We pick $\varepsilon_0 > 0$ so that (1) holds for $n = 0$ and $\varepsilon_0 < \varepsilon$. Suppose f_i, X_i, Y_i, for $i \leq n$, and g_i, for $i < n$, have been constructed. We easily find g_n, X_{n+1}, f_{n+1} and Y_{n+1}, in this order, using Lemma 2.2. Condition (5) can be secured by defining X_{n+1} and Y_{n+1} to be appropriately enlarged $g_n[Y_n]$ and $f_{n+1}[X_{n+1}]$, respectively. Thus, the construction can be carried out.
Fix $n \in \mathbb{N}$ and $x \in X_n$ with $\|x\| = 1$. Using (4) and (1), we get

$$\|f_{n+1} g_n f_n(x) - f_n(x)\| \leq \varepsilon_{n+1} \|f_n(x)\| \leq \varepsilon_{n+1}(1 + \varepsilon_n).$$

Using (1) and (3), we get

$$\|f_{n+1} g_n f_n(x) - f_{n+1}(x)\| \leq \|f_{n+1}\| \cdot \|g_n f_n(x) - x\| \leq (1 + \varepsilon_{n+1}) \cdot \varepsilon_n.$$

These inequalities give

$$(\dagger) \qquad \|f_n(x) - f_{n+1}(x)\| \leq \varepsilon_n + 2\varepsilon_n \varepsilon_{n+1} + \varepsilon_{n+1}.$$

Now it is clear that if the series $\sum_{n\in\mathbb{N}} \varepsilon_n$ converges, then the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy. Let us make a stronger assumption, namely that

$$(\ddagger) \qquad 2\varepsilon_0\varepsilon_1 + \varepsilon_1 + \sum_{n=1}^{\infty} (\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}) < \varepsilon - \varepsilon_0.$$

Given $x \in \bigcup_{n\in\mathbb{N}} X_n$, define $h(x) = \lim_{n\geqslant m} f_n(x)$, where m is such that $x \in X_m$. Then h is an ε_n-isometry for every $n \in \mathbb{N}$, hence it is an isometry. Consequently, it uniquely extends to an isometry on E, which we denote also by h. Furthermore, ($\dagger$) and ($\ddagger$) give

$$\|f(x) - h(x)\| \leq \sum_{n=0}^{\infty} \varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1} < \varepsilon.$$

It remains to see that h is a bijection. To this end, we check as before that $\{g_n(y)\}_{n\geqslant m}$ is a Cauchy sequence for every $y \in Y_m$. Once this is done, we obtain an isometry g_∞ defined on F. Conditions (3) and (4) tell us that $g_\infty \circ h = \mathrm{id}_E$ and $h \circ g_\infty = \mathrm{id}_F$, and the proof is complete. $\square$

3 On universality of the Gurariĭ space

It is known that the Gurariĭ space is isometrically universal among separable Banach spaces. Indeed, as pointed out by Gevorkjan [2], universality follows from the results of Lazar and Lindenstrauss [5] and Michael and Pełczyński [8]: the dual of the Gurariĭ space is a non-separable L_1 space, therefore the Gurariĭ space contains an isometric copy of $C([0,1])$. The reader may also consult the recent paper [1] for another approach.

We conclude with applying our method to proving universality directly, without referring to the structure of the dual or to universality of other Banach spaces.

Lemma 3.1. *Let X_0, X_1, Y_0 be finite dimensional Banach spaces such that $X_0 \subseteq X_1$ and let $f\colon X_0 \to Y_0$ be an ε-isometry, where $\varepsilon > 0$. Then there exist a finite dimensional Banach space Y_1 containing Y_0 and an isometry $g\colon X_1 \to Y_1$ such that*

$$\|g \restriction X_0 - f\| < \varepsilon.$$

Proof. A standard and well known amalgamation property for Banach spaces (already used in the proof of Lemma 2.1 above) says that there exist $W \supseteq Y_0$ and an ε-isometry $f'\colon X_1 \to W$ such that $f' \restriction X_0 = f$. More precisely, $W = (X_1 \oplus Y_0)/\Delta$, where $X_1 \oplus Y_0$ is endowed with the ℓ_1-norm and

$$\Delta = \{(z, -f(z))\colon z \in X_0\}.$$

The space Y_0 is naturally identified with the subspace of W and $f'(x)$ is the equivalence class of $(x, 0)$ (where $x \in X_1$).

Finally, the desired isometry g is provided by Lemma 2.1. □

Theorem 3.2. *Every separable Banach space can be isometrically embedded into the Gurariĭ space.*

Proof. Let $\mathbb{G}$ denote the Gurariĭ space. Fix a separable Banach space X and let $\{X_n\}_{n\in\mathbb{N}}$ be a chain of finite dimensional spaces such that $X_0 = \{0\}$ and $\bigcup_{n\in\mathbb{N}} X_n$ is dense in X. In case X is finite dimensional, we set $X_n = X$ for $n > 0$. We inductively define $f_n\colon X_n \to \mathbb{G}$ so that

(i) f_n is a 2^{-n}-isometry,

(ii) $\|f_{n+1} \restriction X_n - f_n\| < 2 \cdot 2^{-n}$,

for every $n \in \mathbb{N}$. We set $f_0 = 0$. Suppose f_n has already been defined. Let $Y = f_n[X_n]$. Using Lemma 3.1, we find a finite dimensional space $W \supseteq Y$ and an isometry $g\colon X_{n+1} \to W$ such that $\|g \restriction X_n - f_n\| < 2^{-n}$. Using the property of the Gurariĭ space, we find a $2^{-(n+1)}$-isometry $h\colon W \to \mathbb{G}$ such that $h \restriction Y$ is the inclusion $Y \subseteq \mathbb{G}$. Now set $f_{n+1} = h \circ g$. Given $x \in X_n$ with $\|x\| = 1$, we have that $\|g(x) - f_n(x)\| < 2^{-n}$ and hence

$$\|f_{n+1}(x) - f_n(x)\| = \|h(g(x)) - h(f_n(x))\| < (1 + 2^{-(n+1)}) \cdot 2^{-n} \le 2 \cdot 2^{-n}.$$

This shows (ii). Finally, we obtain a sequence $\{f_n\}_{n\in\mathbb{N}}$ that is pointwise Cauchy on each X_n. By (i) and (ii), $f_\infty(x) = \lim_{n\to\infty} f_n(x)$ is a well-defined linear isometry on $\bigcup_{n\in\mathbb{N}} X_n$. This isometry extends uniquely to an isometry $f\colon X \to \mathbb{G}$. $\square$

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References

[1] Avilés, A., Cabello Sánchez, F.C., Castillo, J.M.F., González, M., Moreno, Y., *Banach spaces of universal disposition*, J. Funct. Anal. **261** (2011), 2347–2361

[2] Gevorkjan, Ju.L., *Universality of the spaces of almost universal placement*, Funkcional. Anal. i Priložen **8** (1974), 72 (in Russian); Functional Anal. Appl. **8** (1974), 157 (in English)

[3] Gurariĭ, V.I., *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Sibirsk. Mat. Ž. **7** (1966), 1002–1013 (in Russian)

[4] Kechris, A. S., Pestov, V. G., Todorcevic, S., *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), 106–189

[5] Lazar, A.J., Lindenstrauss, J., *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. **126** (1971), 165–193

[6] Lusky, W., *The Gurarij spaces are unique*, Arch. Math. (Basel) **27** (1976), 627–635

[7] Melleray, J., *Some geometric and dynamical properties of the Urysohn space*, Topology Appl. **155** (2008), 1531–1560

[8] Michael, E., Pełczyński, A., *Separable Banach spaces which admit l_n^∞ approximations*, Israel J. Math. **4** (1966), 189–198

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